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CHROMATIC COLORING OF DISTANCE GRAPHS II

V.YEGNANARAYANAN
Department of Mathematics, Kalasalingam Academy for Research and Education, Deemed to be University, Krishnankoil-626126, TN, India
prof.yegna@gmail.com

M. ANITHA
Department of Mathematics, Kalasalingam Academy for Research and Education, Deemed to be University, Krishnankoil-626126, TN, India
m.anitha@klu.ac.in

NOEMI C. ROHATINOVICI
Polytechnic University of Timişoara and Aurel Vlaicu University of Arad, Arad, Romania Valentina
noemi.rohatinovici@uav.ro

E. BALAS
Department of Automation Industrial Engineering, Textiles and Transport, University, Aurel Vlaicu, Arad, Romania.
balas@drbalas.ro

Abstract
The problem concerning vertex coloring of distance graphs are keenly pursued due to the motivation by the famous Hadwiger-Nelson plane coloring problem(HNP) concerning unit distance graphs. HNP asks for the minimum number of colors required for coloring the points of the two dimensional plane that are separated by a unit distance. In this paper we determine the chromatic number of distance graphs whose distance sets consists of different types of Pythagorean numbers and Semiprime numbers.

Keywords: Distance Graphs, Chromatic number, Pythagorean distance graph, twice Pythagorean distance graph, thrice Pythagorean distance graph, Semiprime distance graph.

1 Introduction

We do not consider graphs that are either non-simple or directed.

Proper vertex coloring of graphs is a vital area of research. Fascinating ideas stems from the issues concerning allotment of channels to several regulating networks such as communication, traffic etc., One can refer [18] for more. In traditional vertex coloring problems [9] a constraint is normally thrust on colors of vertices that are adjacent. But a few other meaningful contexts demand accommodation of constraints such as (i) adjacent vertices and (ii) vertices separated by a distance of two units are colored differently.

For basics and terminology about graphs one can refer to [3, 8]. Proper vertex coloring problem asks for a procedure to allot colors to the vertices such that any pair of adjacent vertices are colored differently. It has a variety of practical applications such as drafting a schedule or time-table, in mobile radio frequency allotment, in suduku, in register allocation, in bipartite graphs, map coloring to name a few. For more such examples one can refer to [6] which is a case study on graph coloring applications.

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1.1 Coloring Graphs With Integer Distance Sets

A graph G is said to have chromatic number $\chi(G) = k$ (k is a positive integer) when there is a function f: $V(G) \to \{1,..., k\}$ that assigns different values to adjacent vertices and k is smallest with this feature. By an integer distance graph G(Z, D) with $D \subseteq Z^+$ we mean the graph with V(G(Z, D)) = Z and any two u, $v \in V(G)$ is said to form an edge iff $|u-v| \in D$. Eggletonet.al in [7] first considered such graphs. A large number of papers were written on this topic, see [7, 21]. For the set of primes P as the distance set the χ(G(Z, D) was computed as4 [4]. The χ(G(Z, D) is completely found when D contains at most three elements. Clearly $\chi(G(Z, D)) = 2$ if D is a one element set. $\chi(G(Z, D)) = 2$ if D contains only odd elements. This is so as |D| +1 bounds $\chi(G(Z, D))$ from above. If D is finite then $\chi(G(Z, D)) = 3$ if D includes two coprime elements of distinct parity. Also one can see [25, 24, 27, 22].

1.2 Pythagorean Triples And Quadruples

Let us denote in what follows a Pythagorean as p. Let Γ(a, b, c) with $a^2+b^2=c^2$ be a p triangle where (a, b, c) is referred as a p triple. If these numbers satisfies (a, b) =1 or (b, c) =1 or (c, a) = 1, then they are called as primitive. Note that if one of (a, b) =1 or (b, c) =1 or (c, a) = 1, then the other two hold good effortlessly.

It is well known that, $a=2qr, b=q^2-r^2, c=q^2+r^2$, or $a=q^2-r^2, b=2qr, c=q^2+r^2$represents all primitive p triples wherein (q, r) = 1, r <q and q, r are of different parity. Infact, the converse of this is also true. It is easy to observe that if (a, b, c) are primitive then so is(sa, sb, sc) where $s \in Z^+$. $qr(q^2-r^2)$ is the area of such a triangle. A p integer is the area of a p triangle. In fact Mohanty et.al in [21] coined the term p integer. s^2x for $r \in Z^+$ is a p number whenever x is so. However the converse need not be true. For example, $60 = 2^2 \times 15$ is a p number, but 15 is not a p number. But $96 = 2^2 \times 24$ is a p number, and 24 is also a p number.

A p quadruple is an ordered 4-tuple (x, y, z, w) such that $x^2 + y^2 + z^2 = w^2$. Suppose that $x^2 + y^2 = k$ and $w = z + \alpha$ then $k + z^2 = (z + \alpha)^2$. This implies that $z = (k - \alpha^2)/2\alpha$. Notice that if k is even then α must be even and if k is odd then α must be odd for $z \in Z$; If k is even then it has to be an integer multiple of 2α.$k > \alpha^2$ is necessary for z to be positive. In order to construct distance graphs whose distance set consists of p quadruples it is pertinent to generate them. For more see [1].

Let x be even and y be odd (alternately one can let x as odd and y as even). In this instance it is clear that k is odd andhence α is odd. If x and y have common factors p_j, j = 1 to n then $k=\prod_{j=1}^{n} p_j^{m_j} \prod_{j=1}^{N} q_j^{s_j}$ and $\alpha = \prod_{j=1}^{n} p_j^{r_j} \prod_{j=1}^{N} q_j^{t_j}$ where m_j, s_j, r_j and t_j all in Z for all j. $z=(k-\alpha^2)/2\alpha$ becomes $z = \frac{1}{2}\left(\prod_{j=1}^{n} p_j^{m_j - r_j} \prod_{j=1}^{N} q_j^{s_j - t_j} - \prod_{j=1}^{n} p_j^{r_j} \prod_{j=1}^{N} q_j^{t_j}\right)$.So with $k > \alpha^2$, t_j can have all integral values from 0 to s_j so that α takes the above form with either $r_j = 0$ or $r_j = m_j$ for all j. For example, if x = 12, y = 15 then $k = 369 = 3^2 \times 41$. Note that we have

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either $\alpha = 1$ or 3^2 andnot 41 as $41^2 > k$. So z = (369–1)/2 = 184 and w = 185 with$\alpha = 1$ or (369 –81)/18 = 16 and w = 25 with $\alpha = 3^2$.So the primitive quadruples for x = 12, y= 15 are (12, 15, 184,185) and (12, 15, 16, 25). Similarly for x = 210, y = 135 we get four different p quadruples (210, 135, 1162, 31163), (210,135, 3458, 3467), (210, 135, 1234, 1259) and (210, 135, 26,251).

Let x and y be both even. Here k should be an even number and hence it follows that s is even. The arguments as in the above instance applies with the exception that some power of 2 arisehere. That is $k = 2^m \prod_{j=1}^{n} p_j^{m_j} \prod_{j=1}^{N} q_j^{s_j}$ and $\alpha = 2^r \prod_{j=1}^{n} p_j^{r_j} \prod_{j=1}^{N} q_j^{t_j}$ .Now z =(k $-\alpha^2)/2\alpha$ implies $z = 2^{(m-r-1)}\left(\prod_{j=1}^{n} p_j^{m_j - r_j} \prod_{j=1}^{N} q_j^{s_j - t_j} - 2^{r-1} \prod_{j=1}^{n} p_j^{r_j} \prod_{j=1}^{N} q_j^{t_j}\right)$.Here the stipulations for getting primitive quadruples are same with an additional stipulation that r = l or r = m–1. So the values of α are$2^l \left(\prod_{j=1}^{n} p_j^{r_j}\right)\left(\prod_{j=1}^{N} q_j^{t_j}\right)$ or$\alpha = 2^{(m-1)}\left(\prod_{j=1}^{n} p_j^{r_j}\right)\left(\prod_{j=1}^{N} q_j^{t_j}\right)$. t_j can take any integer value from 0 to s_j when $k > \alpha^2$ and $r_i = 0$ or n_i. For instance, if x = 6 = 2.3,y = 30 = 2.3.5 then k = 2^3. 3^2. 13 = 936. The various quadruples that results out are: (6, 30, 233, 235), (6, 30, 115, 119), (6, 30,17, 35),(6, 30, 5, 31).

Let x = 2a+1 and y = 2b+1 where a,b$\in Z^+$. Then $k = x^2 + y^2 = (2a+1)^2 + (2b+1)^2$. So $k = 4(a^2+b^2) + 4(a+b) + 2$. k even implies α should be even but k has to contain a factor of 4 to be an integral multiple of 2α.However k is a multiple of 2 only. Hence we infer that no set of quadruples can be got when x and y are both odd.

2 Main Results

If D comprises only positive integers and r$\in Z^+$. If D/gcd(D)contains no multiple of r then $\chi(G(Z, D)) \leq r$. Suppose gcd(D)= 1 then D contains no multiple of r. In this case we can color the vertices of G with colors from 0,1,...,r–1 by allotting to every integer i the color corresponding to the residue class of i modulo r. Two integers will be given same color only when their difference is a multiple of r. As no multiple of r lies in D it results in a chromatic coloring of G(Z, D). The converse of this is also true when r = 2. That is G(Z, D) is bipartite when D/gcd(D) includes integers that are not multiples of 2. It is a fact that if $D \subseteq P$ with $2 \leq |D|$ then $\chi(G(Z, D)) = 2$ if $2 \notin D$, else $\chi(G(Z, D)) = 3$ or 4. Similarly if $2 \in D$ and $3 \notin D$ then $\chi(G(Z, D)) = 3$;if $\{2, 3, 5\}$ or $\{2, 3, 11, 13\} \subseteq D$ then $\chi(G(Z, D))= 4$. So we deduce that:

Theorem 1: Suppose that a distance graph G(Z, D) whose distance set D of cardinality four has only primitive p quadruples and $2 \notin$D. Then $\chi(G(Z, D)) = 2$.

The Theorem 1 is derived out of the observation of the nature of quadruples that are primitive p. None of them have 2 init. if $3 \leq r$ then it is natural ask whether $\chi(G(Z, D)) \leq r$ for finite sets D is NP complete.

Fact 1: Let all the elements in D have x as a common divisor. Then G(Z, D) consists of subgraphs that are disjoint and each isomorphic to G(Z, D/x}) with D/x = {d/x:d $\in$ D}.

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Also this shows that $\chi(G(Z, D)) = \chi(G(Z, D/x))$. In view of this one can deem that gcd(D) =1. [24]

Theorem 2: Deem that (a, b) = 1 when (a, b, c) forms a p 3-tuple. Then (b, c) = 1 and (c, a) = 1.

Proof One can assume either, $c^2 = a^2 + b^2$ or $b^2 = a^2 + c^2$ or $a^2 = c^2 + b^2$ Let $c^2 = a^2 + b^2$. Suppose that b, c$\neq$ 1 and a, c$\neq$ 1. Then there exist $m_1, m_2 \geq 2 \in N$ such that m_1/b, m_1/c and m_2/c, m_2/a. That is $b = m_1t_1$, $c = m_2t_3$ and $a = m_2t_4$ for some t_1, t_3 and t_4. Now $c^2 = a^2 + b^2 \Rightarrow m_2^2t_4^2 + m_1^2t_1^2 = m_2^2t_3^2 = (m_2t_3)^2$, a contradiction as $m_2^2t_2^4 + m_1^2t_1^2$ is not perfect square for any m_1, m_2, t_1 and t_4.

Theorem 2 is not true if a, b, c is not a p triplet. To assert, let (a, b, c) = (2, 3, 4). Then (2, 3) = 1. A positive integer n is p if and only if it has at least 4 distinct positive divisors p, q, r and s such that pq = rs = n and p+q = r–s. [21] Bert Miller [2] has defined a positive integer $n \in Z^+$ as a nasty integer if it has at least four different factors p, q, r and s such that p+q = r–s and pq = rs = n. So n is a nasty integer if and only if it is p. It is known from [21] that there are infinitely many p numbers and every p number is a multiple of 6.We also infer from [21] that if n is p thenr^2n is p for all r. If n and rn are both p then rn is p for every positive integral exponent r^s. So $5^s \times 6$, $2^s \times 6$, $7^s \times 30$ are p for all $s \in Z^+$.

A natural number n is called a twice or a thrice p number if it can be the area of two or three different p triangles. There are an infinite number of twice p numbers. The number 840 is a thrice p number as it is thearea of three p triangles with sides (40, 42, 58), (70, 24, 74) and (112,15,113). So $840x^2$ is a triply p for every $x \in Z^+$. The numbers 210, 2730 and 7980 are the only twice p numbers below 10,000. 210 is the area of two primitive p triangles with sides (12,35,37) and(20,21,29); 2730 for (28,195,197) and (60,91,109); 7980 for(40,399,401) and the following are the 150 p numbers below10,000 [21].

A = {6, 24, 30, 54, 60, 84, 96, 120, 150, 180, 210, 216, 240, 270, 294, 300, 336, 384, 480, 486, 504, 540, 546, 600, 630, 720, 726, 750, 840, 864, 924, 960, 990, 1014, 1176, 1224, 1320, 1344, 1350, 1386, 1470, 1500, 1536, 1560, 1620, 1710, 1716, 1734, 1890, 1920, 1944, 2016, 2100, 2160, 2166, 2184, 2310, 2400, 2430, 2520, 2574, 2646, 2730, 2880, 2904, 2940, 2970, 3000, 3024, 2036, 3174, 3360, 3456, 3570, 3630, 3696, 3750, 3840, 3900, 3960, 4056,4080, 4116, 4290, 4320, 4374, 4500, 4536, 4620, 4704, 4860, 4896, 4914, 5016, 5250, 5290, 5376, 5400, 5544, 5610, 5670, 5766, 5814, 5880, 6000, 6090, 6144, 6240, 6480, 6534, 6630, 6750, 6804, 6840, 6864, 6936, 7140, 7260, 7350, 7440, 7560, 7680, 7776, 7854, 7956, 7980, 8064, 8214, 8250, 8316, 8400, 8640, 8664, 8670, 8736, 8820, 8910, 8976, 9126, 9240, 9360, 9600, 9690 and 9720}.

In the above list, 43 are primitive p numbers.

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B={6, 30, 60, 84, 180, 210, 330, 504, 546, 630, 840, 924, 990, 1224, 1320, 1386, 1560, 1710, 1716, 2310, 2340, 2574, 2730, 3036, 3570, 3900, 4080, 4290, 4620, 4914, 5016, 5640, 5814, 6090, 6630, 7140, 7440, 7854, 7956, 7980, 8970, 8976 and 9690}.

In view of Fact 1, χ(G(Z, B)) = χ(G(Z, B_1))where B_1= {1, 5, 12, 14, 30, 35, 55, 84, 91,105, 140, 154, 165, 204, 220, 231, 260, 285, 286, 385, 390, 429, 455,506, 595, 650, 680, 715, 770, 819, 836, 935, 969, 1015, 1105, 1190,1240, 1309, 1326, 1330, 1495, 1496, 1615}. Walther [26],showed that χ(G(Z, D))≤|D|+1. In view of this, we deduce that χ(G(Z, B_1)) ≤ 44. Yegnanarayanan [28] showed that if D comprises some elements of Z, then χ(G(Z, D)) ≤ min{n$\in$ N : n ($|D_0^n|$+1)} (where D_0^n is the subset of D built by integers divisible by n). In view of this, we calculate the upper bound for various values of n withrespect to the distance set B_1. For n = 1, we get n($|B_1^n|$+1) = 1 $\times$ (43+1) = 44; For n = 2, B_1^2 = {$d_i \in$ B_1:2|d_i i=1, ... 43} = {12, 14, 30, 84, 140, 154, 204, 220, 260, 286, 390, 506, 650, 680,770, 836,1190,1240, 1326,1330,1496}.So, n($|B_1^2|$+1)=2$\times$ (21+1)=44; For n=3, B_1^3= {$d_i \in$ B_1:3|d_ii=1,... 43} = {12, 30, 84,105,165, 204, 231, 285, 390, 429, 819, 969,1326}.So, n($|B_1^3|$+1)=3$\times$(13+1)=42; For n = 4, B_1^4 = {d_i $\in$ B_1 :4|d_i i=1,...,43} = {12, 84,140, 204, 220, 260, 680, 836,1240,1496}. So, n($|B_1^4|$+1) = 3$\times$ (10+1) = 44; For n = 5, B_1^5= {d_i $\in$ B_1: 5|d_i I = 1,...,43} = {5, 30, 35, 55,105,140, 165, 220, 260, 285, 390, 455, 595, 650, 680, 715, 770, 935,1015,1105,1190,1240,1330,1495,1615}.So, n($|B_1^5|$+1)=5$\times$ (23+1) =120; For n= 6, B_1^6={ $d_i \in$ B_1: 6| d_i I = 1,... 43}={12, 30, 84, 204, 390}. So, n($|B_1^6|$ +1) = 6$\times$ (5+1)=36; For n = 7, B_1^7= {d_i $\in$ B_1: 7|d_i i=1,...,43}= {14, 35, 84, 91,105,140,154, 231, 385, 455, 595,770, 819,1015,1190,1309, 1330}. So, n($|B_1^7|$+1)=7$\times$ (17+1) = 126; For n = 8, B_1^8 = {d_i $\in$ B_1: 8|d_i i=1,..., 43} = {680,1240,1496}. So, n($|B_1^8|$+1) = 8$\times$ (3+1) = 32; For n = 9, B_1^9={$d_i \in B_1$: 9|d_i i=1,...,43} = {819}. So, n($|B_1^9|$+1) =9 $\times$ (1+1)=18; Obviously, n=10,11,12,13,14,15 will not yield a better upper bound. But for n=16, we see that B_1^{16}= {$d_i \in B_1$:16|d_i i=1,...,43} = ϕ. So, n($|B_1^{16}|$+1)=16$\times$(0+1)=16; We can stop here as we go higher it is not going to improve. It is clear that $\min_{n\in N}$n($|B_1^n|$+1)=16. Therefore χ(G(Z,B_1)) ≤ 1.

In [23] Voight has established that if D = {x, y, x+y} and (x, y)=1, and D comprises at least one even integer and at least one integer divisible by 3,then χ(G(Z, D))=4. In view of this, we search for a subset B_2 of $B_1$with |B_2| = 3, and the elements of B_2 of the form x, y, x+y satisfying the above conditions. Clearly x=14, y=91 & z=105 will produce such a set. So choose B_2 ={14, 91, 105}.Then χ(G(Z, B_2))=4. Now as $B_2 \subseteq B_1$ and χ is a monotone function, we deduce that χ(G(Z, B_2)) ≥ 4.

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Fact 2: Let D $\subseteq Z^+$and consider G(Z, D). Then G(Z, D) is connected if and only if (d_1, d_2,...) =1 for $d_i \in$ D, i =1, 2,... . This is because there will be a path connecting vertices sand s+1 if and only if $\exists$ d_i, $e_j \in$ D, i=1 to t_1, j=1 to t_2 such that $\sum_{i=1}^{t_1} d_i - \sum_{j=1}^{t_2} e_j = 1$. This can occur only when (d_1, d_2,...) = 1 for all d_i. In view of the above, we derive that:

Theorem 3: Suppose that D is a set of primitive p numbers less than10,000. Then 4 $\le \chi$(G(Z, D)) $\le$ 16.

An alternate proof for the upper bound of Theorem 3 can also be obtained as follows. Consider the set D of positive integers and let r be any positive integer. Derive a new set D_1 from D by dividing each element of D by the greatest common divisor of D. If D_1contains no multiple of r, then χ(G(Z, D_1)) $\le$ r. This is because by Fact 2, we can deem that gcd(D_1)=1. So, we can assign colors 0,1,... r–1or (1,2,... r) to the vertices of G(Z, D_1) by allotting to each integer i the color with respect to the residue class of i modulo r. It is easy to note that two different integers will be allotted to the same color class exactly when the difference is a multiple of r. As no such multiple of ris present inD_1, this results in a proper r-coloring for the vertices of G(Z, D_1). In our case, choose D = B, D_1 = B_1 and r = 16. Then χ(G(Z, B_1)) $\le$ 16.

Since G(Z,B_1)$\subseteq$G(Z,A_1) whereA_1 is obtained from A by reducing all the elements of A upon dividing them by 6 and χ is a monotone function, we have χ(G(Z,B_1)) $\le\chi$(G(Z,A_1)). Note that A_1={1, 4 ,5, 9, 10, 14, 16, 20, 25, 30, 35, 36, 40, 45, 49, 55, 56, 64, 80, 81, 84, 90, 91,100, 105,120,121,125,126,140,144,154,160,165,169,180,196, 204, 220, 224, 225, 231, 245, 250, 256, 260, 270, 285, 286, 289, 315, 320, 324, 336, 360, 361, 364, 385, 390, 400, 405, 420, 429, 455, 461, 480, 484, 490, 495, 500, 504, 506, 529, 560, 576, 595, 605, 616, 625, 640, 650, 660, 676, 680, 686, 715, 720, 729, 50, 756, 770, 784, 810, 816, 819, 836, 841, 845, 875, 880, 896, 900, 924, 935, 945, 961, 969, 980, 1000, 1015, 1024, 1040, 1080, 1089, 1105, 1125, 1134, 1140, 1144, 1156, 1190, 1210, 1225, 1240, 1260, 1280, 1296, 1309, 1326, 1330, 1344, 1369, 1375, 1386, 1400, 1440, 1444, 1445, 1456, 1470, 1485, 1495, 1496, 1521, 1540, 1560, 1600, 1615, 1620}. We find that 41 = $\min_{n\in N}$ n($|A_1^4|$+1) as $|A_1^4|$ = ϕ and hence χ(G(Z, A_1)) $\le$ 41. As χ(G(Z, B_1)) $\ge$ 4, we derive:

Theorem 4: Let D be a set of p numbers less than 10,000. Then 4 $\le \chi$(G(Z, D)) $\le$ 41.

Theorem 5: Let D be a set of twice p numbers less than10,000. Then χ(G(Z, D))=3.

Proof We know that D = {210, 2730,7980}. By Fact 1, χ(G(Z, D)) = χ(G(Z, D_1)) where D_1={1,13, 38}.Note that 210 = gcd(210, 2730, 7980). Clearly gcd(D_1)=1. Kemnitz and Kolberg [13], [14] have showed that if D = {d_1, d_2, ...} is finite and gcd(D)=1, then χ(G(Z, D))=2 if all d_i are odd; χ(G(Z, D))$\le$3 if no $d_i \in$ D is divisible by 3 and χ(G(Z, D)) = 3 if no $d_i\in$ D is divisible by 3 with at least one $d_i \in$ D as even. In view of this, if |D|=r with gcd(D)=1 and if D comprises at least one even integer, then D includes both even and

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odd integers. So G(Z, D) will include in it odd cycles and therefore $\chi(G(Z, D)) \geq 3$. Also by a result of Walther [26] we have $\chi(G(Z, D)) \leq |D|+1 = r+1$. Applying this observation toour distance set D_1 we derive that $\chi(G(Z, D)) = 3$.

Theorem 6: Suppose that D is a set of thrice p numbers less than10,000. Then $3 \leq \chi(G(Z, D)) \leq 4$.

Proof We know that D = {840, 3360, 7560}. As in Theorem 6, we have $\chi(G(Z, D)) = \chi(G(Z, D_1))$ where D ={1, 4, 9}. Once again making use of the observations in Theorem 6, we deduce that $3 \leq \chi(G(Z, D_1)) \leq 4$.

p triples were found in Babylon in the form of cuneiform tablets and are made use of in vedic rituals. They are also mentioned in early geometric books of India. A foremost mention of Pythagoras theorem can be seen in a text of Baudhayana about 800 B.Cin Sulbasutra [19]. For more one can also see ([5], [12], [10,16,17,19,20]). Infact, it would be interesting to note that p triples can be represented as Gopala-Hemachandra numbers. Consider the Gopala-Hemachandra G+1 quadruple (s, t, u, v). The G+1 sequence was attributed to those Indian Mathematicians who belonged to the period prior to Fibonacci[28]. It is given as s, t, s+t, s+2t, 2s+3t, 3s+5t,... for any pair s, t. When s = t= 1, we derive the Fibonacci sequence. In the G+1 quadruple (s, t, u, v) if m_1 = st, $m_2 = u(m_3 - m_1)/t$ and m_3 =tv+us, then (m_1, m_2, m_3) is a p triple. If the quadruple (s, t, u, v) has no common divisors and s is odd, then (m_1, m_2, m_3) is a primitive p triple. The values m_2 and m_3 are shown as m_2 = 2tu, $m_3 = t^2+u^2$. This shows that m_1 can be got by multiplying s and v. For example, the G+1 quadruple (1, 1, 2, 3) results in $m_3 = 1^2+2^2 =5$, $m_1= 1 \times 3$ and $m_2=2 \times 2=4$. The G+1 quadruple can be visualized as $(b, \frac{(a-b)}{2}, \frac{(a+b)}{2}, a)$ where a, b are distinct odd integers and a > b.

Theorem 7: Let D be a set of cardinality 3 comprising thrice p numbers. Then $\chi(G(Z, D)) \geq 3$.

ProofIt is known that 840 is a p number and $840x^2$ is a thrice p number for all x belonging to Z^+. Now choose the distance set D to consist of all those square multiples of 840 which forms a p triple. For instance, once such D can be D = $\{840 \times 3^2, 840\times 4^2, 840 \times 5^2\}$. Then by Fact 1 we can consider that D_1 obtained from D by dividing each of its elements by 840. That is $D_1 =\{a^2, b^2, c^2\}$ where (a, b, c) forms a p triple. Then as $\chi(G(Z, D))=\chi(G(Z, D_1))$, we compute the chromatic number of $\chi(G(Z, D_1))$. Note that subgraph $G(Z, D_2)$ of the graph $G(Z, D_1)$ on the vertex set $D_2 = \{0^2, b^2, c^2\}$ is isomorphic to K_3 as $a^2-0^2 = a^2$, $c^2-0^2 = c^2$ and $c^2-a^2 = (m^2+n^2)^2-(m^2-n^2)^2 = 4m^2n^2 = (2mn)^2 = b^2$ is a p triple of D_1. This shows that $\chi(G(Z, D_2))$ =3 and as f is a monotone function $\chi(G(Z, D_1)) \geq 3$. This implies that $\chi(G(Z, D)) \geq 3$.

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Theorem 8: Let D be a set of distinct quadruples of the form {a, b, $\frac{(a-b)}{2}$, $\frac{(a+b)}{2}$,} where a < b, a and b are distinct odd integers and gcd(a,b)=1. Then $3 \leq \chi(G(Z, D)) \leq 5$

Proof Note that as a and b are both distinct odd integers, it is trivial that $\frac{(a-b)}{2}$ and $\frac{(a+b)}{2}$, are even. Therefore by Lemma 9 of [19], [20] we infer that $3 \leq \chi(G(Z, D)) \leq 5$.

Theorem 9: Suppose that D = {a, b, $\frac{(a+b)}{2}$, $\frac{(b-a)}{2}$} with a < b are a set of distinct G+1 quadruples with gcd(a,b)=1. If a =1 and $b \equiv 1 \pmod 2$ then $\chi(G(Z, D)) \geq 4$.

Proof Suppose that it is possible to color the vertices of Z with colors under the hypothesis of the theorem. Let g: V(G(Z,D)) $\rightarrow$ {p, q, r, s} be one such coloring. As a =1 and $b \equiv 1 \pmod 2$, the distance set D has its elements as D = {1,k,k+1,2k+1} where $k \geq 2$. It is easy to see that k = 1 is excluded as the elements are distinct. As 0,k, k+1 are to be assigned with different colors, let us take g(0) = p, g(k) = q and g(k+1) = r. This means g(1) can be either q or s as 1 is adjacent with both 0 and k+1. If g(1) = q, then g(2k+1) = s as 2k+1 is adjacent with 0, k and k+1. If g(1) = s, then g(2k+1) can be assigned the color of 1. That is g(2k+1) =s. Now consider the vertex 2k+2. Clearly g(2k+2) = p or q as 2k+2 is adjacent with both k+1 and 2k+1. If g(2k+2) = p then g(k+2) = s as k+2 is adjacent with 1, k+1, 2k+2 with colors q, r and p. Note that the vertices 1,k+1, k+2, 2k+2 with respective colors q,r,s,p induces a K_4 in G(Z, D). Therefore $\chi(G(Z, D)) \geq 4$.

Note 1: Suppose that $x \in Z$ be any arbitraryvertex of G(Z,D). For G(Z, D) to contain a K_5, x has to be adjacent with vertices 1, k+1, k+2, 2k+2, i.e. x should be one of 2, k+2, k+3, 2k+1, 2k+2, 2k+3, 3k+2, 3k+3 or 4k+3. Clearly $x \neq$ k+1, k+2. Further it is not difficult to rule out the rest of the possibilities. However it is not known whether any other graph requiring five colors could be a subgraph of G(Z,D). Hence we raise the following conjecture.

Conjecture 1: $\chi(G(Z,D)) = 4$ where D comprises distinct G+1 quadruples with D = {1, k, k+1, 2k+1} for $k \in Z^+$.

We call two p numbers as twin p numbers if the difference of the larger and the smaller is six. The following are the twin p numbers less than 10,000. (24, 30), (54, 60),(210, 216), (330, 336), (480, 486), (540, 546), (720, 726), (750, 756),(1710,1716), (2160, 2166), (8664, 8670) and (8970, 8976).

Theorem 10: Suppose that D ={x, y} where x and y are twin pnumbers. Then $\chi(G(Z, D)) = 3$.

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Proof We know by a result of Kemnitz and Kolberg [13], [14] that if |D| =r with gcd(D)=1 and if D comprises atleast one even integer then $\chi(G(Z, D))\geq 3$. Also by a result of Walther [26] we have $\chi(G(Z, D)) \leq |D|+1 = r+1$. As gcd(x, y) =1, if x and y are twin p numbers and D comprises at least one even number, we conclude that $\chi(G(Z, D)) = 3$ as it satisfies the conditions of the results of Kemnitz et.al [13], [14] and Walther. [26]

Suppose that $n\in Z^+$ and p is a prime. If $p^2| n$ whenever p | n then we call such an n, a powerful number. A p powerful number is one which is both powerful and p. Note that if n is p then the number s^2n^{2t+1} is a powerful p number for all s, $t\in Z^+$. The number 216 is the least powerful p number. Some sample powerful p numbers are listed as $s^2\times 6^3$; $s^2\times 2^m\times 30^3$; $s^2\times 5^m\times 6^3$; $s^2\times 7^m\times 30^3$. If D is a collection of powerful p numbers then D is infinite and after cancelling the common factor among them it may look like 1; $2^m\times 5_3$;5^m; $7^m\times 5^3$,... Clearly gcd(D)=1 and it contains at least one even integer. So $\chi(G(Z, D)) \geq 3$ and $\chi(G(Z, D)) \leq |D| +1$.

Problem: Determine $\chi(G(Z, D))$ when D is a collection of powerful p number.

3 Semiprimes

A product of two prime numbers form a semiprime number. Semiprimes have applications in security, authentication and identification. For example, the RSA algorithm in public-key cryptography. When n is large, it is very difficult to find its factors p and q and all best known algorithms run in exponential time. This feature of asymmetry is responsible for being used as public keys in cryptography. The digital certificate employed to run programs on gridhad a 2048-bit modulus semiprime n. Note that n is public and found in both public and private key. So its security relies on the ability to find its prime factors. Once its factors are known, it is easy to generate the corresponding private key and hence breach its security. Advanced techniques like elliptic curve factorization could not consume less time in the decomposition process for semiprimes of 50 to 100 digits length.

Huge primes and semiprimes have the form 6s + 1 or 6s –1, where sis a positive integer. So the total number of primes lying between p_1 and p_2 is exactly the number of 6n+1 or 6n–1 primes lying in that range. Hence any semiprime n_0 is one of: $n_0 = (6n+1)(6m+1)$, $n_0 = (6n-1)(6m-1)$ and $n_0 = (6n-1)(6m+1)$.

We now find the χ of a graph whose distance set comprises semiprimes of the following four types: When the distance set A_i, i= 0,1,2,3 is a set of finite semiprimes congruent to i (mod 4) when i = 0,1,2,3. The chromatic number happens to be either 2 or 3. The trend of distribution of the semiprime pattern indicates that the chromatic number remains unaffected even when the three different distance sets A_i, i = 1, 2, 3get enlarged with either finite or infinite number of respective semiprimes. This observation leads to Conjecture 2.

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Suppose that A_0=4; A_1 = {9, 21, 25, 33 ,49, 57, 65, 69, 77, 85, 93,121,129,133,141,145,161,169,177,185,201, 205, 209, 213, 217, 221, 237, 249, 253, 265, 289, 301, 305, 309}; A_2 = {6,10,14, 22, 26, 34, 38,46, 58, 62, 74, 82, 86, 94,106,118,122,134,142,146,158, 166,178,194, 202, 206, 214, 218, 226, 254, 262, 274, 278, 298, 302, 314}; A_3 = {15, 35, 39, 51, 55, 87, 91, 95,111,115,119,123,143,155,159,183,187, 203, 215, 219, 235, 247, 259, 267, 287,291, 295, 299, 303};

The elements of A_i are congruent to i (mod 4) for I = 0,1, 2, 3. As 2 is the only even prime it is easy to see that 4 is the only semi prime congruent to 0(mod 4) and A_0will be always a singleton set. Let us now determine the chromatic number of G(Z, D=A_1). We know that χ(G(Z, D=A_1)) ≤ $\min_{n \in N}$ n($|D_0^n|$ + 1) [28] where D_0^n is the subset of D built by integers divisible by n. Using this we now calculate the upper bound forvalues of n with respect to distance set D = A_1.

For n=1, we get n($|\mathcal{A}_1^n+1|$)=1(34+1)=35 For n = 3,we get 3($|\mathcal{A}_1^3+1|$)=3(15+1)=48 For n = 5, we get 5($|\mathcal{A}_1^5+1|$) = 5(8+1) = 45 For n = 7, we get7($|\mathcal{A}_1^7+1|$)=7(5+1)=42 For n = 9, we get 9($|\mathcal{A}_1^9+1|$) = 9(1+1) = 18,... One can deduce that χ(G(Z, A_1)) ≤ 18. But from the result of Kemnitzand Kolberg [13.14] discussed earlier in Theorem 7, we find that as gcd(A_1) = 1, χ(G(Z, A_1)) = 2 as all elements of A_1 are odd. The same is the case with A_3 also. Therefore χ(G(Z, A_3)) = 2. In view of this we state:

Theorem 11: If D = A_1 or A_3 thenχ(G(Z, A_1) = χ(G(Z, A_3)) = 2.

Also from one of the results of Kemnitz and Kolberg [13,14] discussed inTheorem 7, we derive the following result as the elements of A_2 are not congruent to 0 (mod 3) and has at leasteven integer.

Theorem 12: If D = A_2 thenχ(G(Z, D= A_2) = 3.

Corollary 1: χ(G(Z, A_0)) = 2.

Conjecture 2: Find χ(G(Z, D)) where D=$\{A_i^*\}$ and the elements of A_i are congruent to i (mod 4), i=1, 2, 3 is (a) a finite set of semiprimes and (b) an infinite set of semiprimes.

Conclusion

Motivated by the results of Eggleton we have computed both the lower and upper bounds for the χ of certain distance graphs whose distance set elements are either primitive p numbers or twice p numbers or thrice p numbers or p quadruples. Then stimulated by the usefulness of semiprimes in cyber security we determined the chromatic number of certain distance graphs with distance set elements as semiprimes. Our results may find applications in resolving issues related to interference graphs as its chromatic number determination plays a vital role and we hope torevert back on this aspect elsewhere.

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